Nonhomogeneous PDEs with Nonhomogeneous Boundary Conditions

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Abstract

Exercise 4.4, 10

1 General Idea

First, we need to transform the Nonhomogeneous Boundary Conditions to homogenous ones in order to use separation of variables.

1.1 Nonhomogeneous Boundary Conditions

Considering a heat problem

$$u_t = \alpha^2 u_{xx} \tag{1}$$

with initial condition

$$u(x,0) = f(x) \tag{2}$$

and boundary conditions

$$u(0,t) = T_1,\ u(L,t) = T_2 \tag{3}$$

The idea is to transform the problem into a homogenous problem. Todo this, we want to find a function such that $u(x,t) = w(x,t) + v(x)$ with one condition: $v_t = v_{xx} = 0$. This will require v to be liner in x and independent of t. Applying the initial conditions (3), we can solve for the new coefficients of $v(x)$:

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$$w(0,t) = u(0,t) - c_2 = T_1 - c_2 = 0$$
$$w(L,t) = u(L,t) - (c_1 L + c_2) = T_2 - c_1 L - c_2 = 0$$

Our new unknown w is given by

$$w(x,t) = u(x,t) - \frac{T_1 + T_2}{L}x - T_2$$

with new initial conditions:

$$w(x,0) = f(x) - \frac{T_1 + T_2}{L}x - T_2$$

Therefore, our new homogenous problem will look like this:

$$w_t = \alpha^2 w_{xx} \tag{4}$$
$$w(x,0) = f(x) - \frac{T_1 + T_2}{L}x - T_2 \tag{5}$$
$$w(0,t) = w(L,t) = 0 \tag{6}$$

Remeber, the solution to the original Problem will be given by

$$u(x,t) = w(x,t) + \frac{T_1 + T_2}{L}x + T_2 \tag{7}$$

1.2 Solving Nonhomogeneous PDEs

Considering a heat problem

$$u_t = u_{xx} + F(x,t) \tag{8}$$

with initial condition

$$u(x,0) = f(x) \tag{9}$$

and boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{10}$$

has the homogenous solution

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) sin \frac{n\pi x}{L} \tag{11}$$

Because (8) is not homogenous, we want to find a solution similar to (11) but with a different $T_n(t)$ as the solution to our problem.

1.2.1 Calculation of $T_n(t)$

Expanding F(x,t) into its Fourier sine series for each value of t on its Domain $[0, L]$ we find

$$F(x,t) = \sum_{n=1}^{\infty} F_n sin\frac{n\pi x}{L}dx \tag{12}$$

where F_n is given by

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)sin\frac{n\pi x}{L}dx \tag{13}$$

The key step is to find $T_n(t)$ is to substitute (11) into the PDE (8). Differentiating term by term, we obtain

$$\sum_{n=1}^{\infty} T_n'(t)sin\frac{n\pi x}{L} = -\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^2}T_n(t)sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty}F_n sin\frac{n\pi x}{L} \tag{14}$$

and simplifying, we find that

$$T_n'(t) + \frac{n^2\pi^2}{L^2}T_n(t) = F_n(t)$$

We can find an integrating factor for this ODE, in this case $e^{\frac{n^2\pi^2 t}{L^2}}$, to solve for $T_n(t)$:

$$T_n(t) = e^{\frac{-n^2\pi^2 t}{L^2}}\int_0^t e^{\frac{n^2\pi^2\tau}{L^2}}F_n(\tau)d\tau + k_n e^{\frac{-n^2\pi^2 t}{L^2}},\ n = 1, 2, 3, ..$$

The only thing left is to apply the initial condition (9) to find k_n

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} k_n sin\frac{n\pi x}{L}$$

$$k_n = \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx \tag{15}$$

1.3 Example 1

In this example we considers a nonhomogeneous PDE with a homogenous boundary value problem.

$$u_t = u_{xx} + 10 \tag{16}$$

with initial condition

$$u(x,0) = 3sin(x) - 4sin(2x) + 5sin(3x) \tag{17}$$

and boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0 \tag{18}$$

Calculating eigenvalues for the homogenous equation, we find that our eigenfunction has the form of

$$u(x,t) = \sum_{k=1}^{\infty} T_n(t)sin(nx) \tag{19}$$

Now we need to expand $F(x,t)$ (the nonhomogeneous part of (17)) into its sine Fourier Series. Showing step-by-step procedure which is equivalent to (13):

$$F(x,t) = 10 = \sum_{k=1}^{\infty} F_n sin(nx)$$

$$\int_0^{\pi} sin^2(x)F_n dx = \int_0^{\pi} F(x,t)sin(nx)dx$$

$$F_n = \frac{10\int_0^{\pi} sin(nx)dx}{\int_0^{\pi} sin^2(x)dx}$$

leading to

$$F_n = \frac{20}{n\pi}(1 - cos(n\pi))$$

Now, according to (14), we need to substitute (19) into (16) in order to calculate $T_n(t)$:

$$\sum_{k=1}^{\infty} T_n'(t)sin(nx) = \sum_{k=1}^{\infty} -n^2 sin(nx)T_n(t) + \sum_{k=1}^{\infty} \frac{20}{n\pi}(1 - cos(n\pi))$$

$$T_n'(t) + n^2 T_n(t) = \frac{20}{n\pi}(1 - cos(n\pi))$$

Using an integrating factor $I = e^{n^2t}$, we can solve this ODE

$$(e^{n^2t}T_n(t))' = \frac{20}{n\pi}(1 - cos(n\pi))e^{n^2t}$$

$$T_n(t) = \frac{20}{n^3\pi}(1 - cos(n\pi)) + c_n e^{-n^2t}$$

and therefore our general solution is

$$u(x,t) = \sum_{k=1}^{\infty}[\frac{20}{n^3\pi}(1 - cos(n\pi)) + c_n e^{-n^2t}]sin(nx)$$

Now we need to use our initial conditions given by (17) to solve for the constant c_n in order to get the particular solution for this problem.

$$u(x,0) = 3sin(x) - 4sin(2x) + 5sin(3x) = \sum_{k=1}^{\infty}[\frac{20}{n^3\pi}(1 - cos(n\pi)) + c_n]sin(nx)$$

We notice that these indictional conditions are set up in a convenient way, so we do not need to use a Fourier series but can use values of n directly, namely $n = 1, n = 2$, and $n = 3$ respectively:

$$c_1 = 3 - \frac{20}{\pi}(1 - cos(n\pi) \quad \rightarrow \quad c_1 = 3 - \frac{40}{\pi}$$

$$-4 = \frac{20}{8\pi}(1 - cos(2\pi)) + c_2 \quad \rightarrow \quad c_2 = -4$$

$$5 = \frac{20}{27\pi}(1 - cos(3\pi)) + c_3 \quad \rightarrow \quad c_3 = 5 - \frac{40}{27\pi}$$

and for $n \geq 4$

$$\frac{20}{n^3\pi}(1 - cos(nx)) + c_n = 0 \quad \rightarrow \quad c_n = -\frac{20}{n^3\pi}(1 - cos(nx))$$

We are going to do a little trick here. Since we would like to start our summation from $n = 1$, we need to pull the fractions for $n = 1, 3$ inside so they will be included in the series. Our final solutions looks therefore like this:

$$u(x,t) = 3e^{-t}sin(x) - 4e^{-4t}sin(2x) + 5e^{-9t}sin(3x) + \sum_{k=1}^{\infty}[\frac{20}{n^3\pi}(1 - cos(n\pi)) -$$

$$\frac{20}{n^3\pi}(1 - cos(nx))e^{-n^2t}]sin(nx)$$

1.4 Example 2

In this example we considers a homogeneous PDE with a nonzero boundary value problem.

$$u_t = u_{xx}$$

with initial condition

$$u(x,0) = \frac{10x^2}{2\pi} \tag{20}$$

and boundary conditions

$$u_x(0,t) = 0 \ u_x(\pi,t) = 10 \tag{21}$$

Though this looks like an homogenous equations, it is in fact not since we need to do a transformation first in order to use separation of variables. Notice that $v(x)$ can not be linear anymore since we have a boundary-value-problem and both conditions are giving in the first derivative of u. Assuming that

$$w(x,t) = u(x,t) - c_1x^2 - c_2x - c_3$$

and applying conditions from (21), we find that $c_1 = \frac{10}{2\pi}$, $c_2 = 0$ and $c_3 = 0$. Therefore, we must solve the nonhomogeneous equation in respect to

$$w_t = w_{xx} + \frac{10}{\pi} \tag{22}$$

initial conditions

$$w(x,0) = u(x,0) - \frac{10x^2}{2\pi} = \frac{10x^2}{2\pi} - \frac{10x^2}{2\pi} = 0 \tag{23}$$

and new boundary-values:

$$w_x(0,t) = w_x(\pi,t) = 0 \tag{24}$$

Solving the corresponding homogenous equation by separation of variables (that is we assume $w(x,t) = X(x)T(t)$), we find that

$$X_n = c_n + cos(nx) \ \ T_n = D_n + e^{-n^2t} \quad n = 1, 2, 3, ...$$

We find that our solution has the form

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t) cos(nx)$$

Now, we need to expand $F(x,t) = \frac{10}{\pi}$ into its Fourier cosine series on $0 \leq x \leq \pi$.

$$F_n(t) = \frac{2}{\pi} \int_0^\pi F(x,t) cos(nx) dx = \frac{20}{\pi} \tag{25}$$

Notice that we only get a solution for $n = 0$. We need to obtain a formula to calculate T_n according to (14) and we find that

$$T_n'(t) + n^2 T_n(t) = c_n(t) \tag{26}$$

using the integrating factor $e^{n^2 t}$ we can solve for $T_n(t)$

$$T_n(t) = e^{-n^2 t} \int_0^t e^{n^2 \tau} F_n(\tau) d\tau \tag{27}$$

which is somewhat simpler than the general equations since we expand k_n about the Fourier cosine series of the initial time conditions (23), which is equal to 0, and therefore $k_n = 0$. Evaluating this integral is rather easy since $F_n(\tau)$ does not explicitly depend on τ.

$$T_n(t) = \frac{20}{n^2 \pi}(1 - e^{-n^2 t}) \tag{28}$$

Therefore, our final solution is

$$w(x,t) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 t}}{n^2} cos(nx) \quad n = 1, 2, 3, .. \tag{29}$$

The steady state temperature is found for large t, so

$$\lim_{t \to \infty} u(x,t) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{cos(nx)}{n^2} \quad n = 1, 2, 3, ..$$